Acyclic and Star Colorings of Cographs *

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Abstract

An *acyclic coloring* of a graph is a proper vertex coloring such that the union of any two color classes induces a disjoint collection of trees. The more restricted notion of *star coloring* requires that the union of any two color classes induces a disjoint collection of stars. We prove that every acyclic coloring of a cograph is also a star coloring and give a linear-time algorithm for finding an optimal acyclic and star coloring of a cograph. We also show that the acyclic chromatic number, the star chromatic number, the treewidth plus one, and the pathwidth plus one are all equal for cographs.

1 Introduction

A *proper vertex coloring* (or *proper coloring*) of a graph G is a mapping $\phi : V \to \mathbb{N}^+$ such that if a and b are adjacent vertices, then $\phi(a) \neq \phi(b)$. The chromatic number of a graph G, denoted $\chi(G)$, is the minimum number of colors required in any proper coloring of G. An *acyclic coloring* of a graph is a proper coloring such that the subgraph induced by the union of any two color classes is a disjoint collection of trees. A *star coloring* of a graph is a proper coloring such that the subgraph induced by the union of any two color classes is a disjoint collection of stars. The acyclic and star chromatic numbers of G are defined analogously to the chromatic number and are denoted by $\chi_a(G)$ and $\chi_s(G)$, respectively. Since a disjoint collection of stars constitutes a forest, it follows that every star coloring is also an acyclic coloring and $\chi_a(G) \leq \chi_s(G)$ for every graph G. We will find it useful to consider the alternative definitions that result from the following observation.

Observation 1. *Let ϕ be a proper coloring of a graph G.*

ϕ is an acyclic coloring of G if and only if every cycle in G uses at least three colors.

ϕ is a star coloring of G if and only if every path on four vertices in G uses at least three colors.

A great deal of graph-theoretical research has been conducted on acyclic and star coloring since they were introduced in the early seventies by Grünbaum [18]. Our investigation of these problems from an algorithmic point of view is motivated in part by their applications in combinatorial scientific computing, where they model the optimal evaluation of sparse Hessian matrices. In fact, these coloring problems were independently discovered and studied by the scientific computing community. The survey of Gebremedhin et al. [14] gives a history of the subject as well as an overview of the use of these coloring variants in computing sparse derivative matrices.

The acyclic and star coloring problems are both **NP**-hard, and most results concerning their complexity on special classes of graphs are negative. In particular, both problems remain **NP**-hard even when restricted to bipartite graphs [8, 9]. In addition, Albertson et al. [1] showed that the problem of determining whether the star chromatic number is at most three is **NP**-complete even for planar bipartite graphs. The authors also showed that it is **NP**-complete to decide whether the chromatic number of a graph G is equal to the

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star chromatic number of G, even if G is a planar graph with chromatic number three. Inapproximability results for both problems are given in [16].

Researchers have obtained a few positive algorithmic results for these problems on graphs for which the acyclic or star chromatic number is bounded by a constant. In particular, Skulrattanakulchai [23] gives a linear-time algorithm for finding an acyclic coloring of a graph with maximum degree three that uses four colors or fewer, and Fertin and Raspaud [11] give a linear-time algorithm for finding an acyclic coloring of a graph with maximum degree five that uses nine colors or fewer. To our knowledge, prior to this work no polynomial time algorithm was known for either of these problems on a nontrivial class of graphs for which the acyclic or star chromatic number is unbounded.

In this paper, we consider acyclic and star colorings of cographs, which are characterized by the fact that they do not contain an induced path on four vertices [10]. This well-studied class has many other characterizations; see [7, Theorem 11.3.3] for a partial list. Many problems that are **NP**-complete on general graphs have polynomial time algorithms when restricted to cographs, in part because of the nice decomposition properties that these graphs exhibit. Nevertheless, problems such as list coloring and achromatic number remain **NP**-complete on this class [2, 19]. Our motivation, however, stems also from the fact that the cographs can be characterized by the property that every acyclic coloring is also a star coloring. We begin Section 2 with a proof of this fact. We then describe a constructive linear-time algorithm for finding an optimal acyclic and star coloring of a cograph. When G is given as an adjacency list, our algorithm runs in $O(n+m)$ time, where n and m are the numbers of vertices and edges in G, respectively; only $O(n)$ time is required when G is given as a corresponding cotree.

Bodlaender and Möhring [5] showed that the pathwidth of a cograph equals its treewidth. In Section 3, we prove that the acyclic colorings of a cograph G coincide with the proper colorings of triangulations of G. As a consequence, we find that the acyclic chromatic number, the star chromatic number, the treewidth plus one, and the pathwidth plus one are all equal for cographs. Additionally, we discuss the implications of our results for the perfect phylogeny problem on this class of graphs.

2 Cographs

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs such that $V_1 \cap V_2 = \emptyset$. The *disjoint union* of G_1 and G_2 is the graph $G_1 \bigcup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The *join* of G_1 and G_2 is the graph $G_1 * G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup \{v_1 v_2 \mid v_1 \in V_1, v_2 \in V_2\})$.

Definition 1 (cograph)**.** *A graph $G = (V, E)$ is a* cograph *if and only if one of the following conditions holds:*

(i) $|V| = 1$;

(ii) *there are cographs $G_1, \ldots, G_k$ such that $G = G_1 \bigcup G_2 \bigcup \cdots \bigcup G_k$;*

(iii) *there are cographs $G_1, \ldots, G_k$ such that $G = G_1 * G_2 * \cdots * G_k$.*

As noted in Section 1, the cographs are exactly the graphs with no induced path on four vertices.

Theorem 1. *A graph G is a cograph if and only if every acyclic coloring of G is also a star coloring of G.*

Proof. Any path on four vertices P in a cograph must either induce a cycle or contain an induced triangle; thus any acyclic coloring of G must use at least three colors for P.

A graph for which every acyclic coloring is a star coloring cannot contain an induced path on four vertices and therefore must be a cograph. □

Corollary 2. *If G is a cograph, then $\chi_s(G) = \chi_a(G)$.*

In the remainder of this section, we develop a linear-time algorithm for finding optimal acyclic and star colorings of cographs. We begin with a general result concerning the acyclic and star chromatic numbers of graphs formed by the join and disjoint union operations.

Lemma 1. *The following hold for any graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$.*

(i) $\chi_a(G_1 \bigcup G_2) = \max\{\chi_a(G_1), \chi_a(G_2)\}$.

(ii) $\chi_s(G_1 \bigcup G_2) = \max\{\chi_s(G_1), \chi_s(G_2)\}$.

(iii) $\chi_a(G_1 * G_2) = \min\{\chi_a(G_1) + |V_2|, \chi_a(G_2) + |V_1|\}$.

(iv) $\chi_s(G_1 * G_2) = \min\{\chi_s(G_1) + |V_2|, \chi_s(G_2) + |V_1|\}$.

Proof. The proofs of (i) and (ii) are obvious.

Our proof of (iii) begins by showing that $\chi_a(G_1*G_2) \leq \min\{\chi_a(G_1)+|V_2|, \chi_a(G_2)+|V_1|\}$. Let $G \equiv G_1*G_2$. We describe an algorithm that, given optimal acyclic colorings of G_1 and G_2, produces an acyclic coloring ϕ of G that uses the desired number of colors. Let ϕ_1 and ϕ_2 be arbitrary optimal acyclic colorings of G_1 and G_2, respectively. Assume without loss of generality that $\chi_a(G_2) + |V_1| \leq \chi_a(G_1) + |V_2|$. We construct ϕ as follows. Color those vertices in V_2 the same as they are colored by ϕ_2, using the colors in $\{1, \ldots, \chi_a(G_2)\}$. Color those vertices in V_1 such that each $v \in V_1$ receives a distinct color in $\{\chi_a(G_2) + 1, \ldots, \chi_a(G_2) + |V_1|\}$. Suppose that ϕ causes a bichromatic cycle $C \subseteq V$ in G. Since each vertex in V_1 gets a distinct color and no vertex in V_1 shares a color with a vertex in V_2, it follows that $C \cap V_1 \leq 1$. If $|C \cap V_1| = 1$, then C induces an edge between distinct $a, b \in V_2$, which implies that C cannot be bichromatic. Hence C must be contained entirely in V_2, which contradicts the fact that ϕ_2 is an acyclic coloring of the subgraph of G induced by V_2. Thus ϕ is an acyclic coloring of G, which completes this direction of the proof.

Now suppose that $\chi_a(G) < \min\{\chi_a(G_1) + |V_2|, \chi_a(G_2) + |V_1|\}$, and let ϕ be an optimal acyclic coloring of G. Since ϕ is also an acyclic coloring of the subgraphs induced in G by V_1 and V_2, which must receive disjoint sets of colors, it follows that there exist $a_1, b_1 \in V_1$ such that $\phi(a_1) = \phi(b_1)$ and $a_2, b_2 \in V_2$ such that $\phi(a_2) = \phi(b_2)$. Thus $a_1a_2b_1b_2$ is a bichromatic C_4 in G, which is a contradiction.

The proof of (iv) is similar to that of (iii). $\square$

Cographs can be recognized in linear time [10], and most recognition algorithms also produce a special decomposition structure in the same time bound when the input graph G is a cograph. We now introduce this structure, which is often used in algorithms designed to work on cographs. We associate with a cograph G a rooted tree T_G called a *cotree*, whose leaves are in one-to-one correspondence with the vertices of G. For the sake of clarity, we will use the word "node" when referring to cotrees, whereas the term "vertex" will be reserved for the context of the original graph G. For a node t_j in T_G, V_j denotes the set of vertices in G that correspond to leaves in the subtree of T_G rooted at t_j; we denote by G_j the subgraph of G induced by V_j. Every internal node of T_G is labeled either 0 or 1, corresponding to the disjoint union and join operations, respectively, in the following way. Let t_i be an internal node of T_G with children $\{t_1, \ldots, t_k\}$. If t_i is a 0-node, then $G_i = G_1 \bigcup \cdots \bigcup G_k$. If t_i is a 1-node, then $G_i = G_1 * \cdots * G_k$. The *canonical* cotree of a cograph is unique up to isomorphism and has the property that any path from a leaf to the root alternates between 0-nodes and 1-nodes. It is often more convenient to work with cotrees whose internal nodes have exactly two children. Since the operations $\bigcup$ and $*$ are commutative and associative, one can show that any cotree T can, in linear time, be converted into a cotree T' such that T' that meets this condition and has size linear in that of T [5]. We will therefore assume throughout this paper that all cotrees are given in binary form.

An example is shown in Figure 1.

Our algorithm for finding an optimal acyclic coloring of a cograph consists of two phases. In the first phase, we traverse the cotree from the leaves to the root, computing for every tree node t_i the values $|V_i|$ and $\chi_a(G_i)$. Additionally, we mark one child of every 1-node as being *saturated*. These markings will be used in the second phase, where they will indicate that all of the vertices associated with leaves in the corresponding subtree should receive distinct colors.

Theorem 3. *Given a cograph G and a corresponding binary cotree T_G, an optimal acyclic coloring of G can be found in $O(n)$ time. Moreover, the obtained coloring is also an optimal star coloring.*

Proof. We construct the desired coloring as follows. For every leaf node t_ℓ in T_G, initialize $\chi_a(G_\ell) = 1$, and mark t_ℓ as saturated. Now let t_i be an internal node of T_G whose children t_j and t_k have already been visited. If t_i is a 0-node, then set $\chi_a(G_i) = \max\{\chi_a(G_j), \chi_a(G_k)\}$. If t_i is a 1-node, then assume without loss of generality that $|V_j| + \chi_a(G_k) < |V_k| + \chi_a(G_j)$, set $\chi_a(G_i) = |V_j| + \chi_a(G_k)$, and mark t_j as saturated. When we reach the root, we will have computed $\chi_a(G)$. What remains is to construct an acyclic coloring of

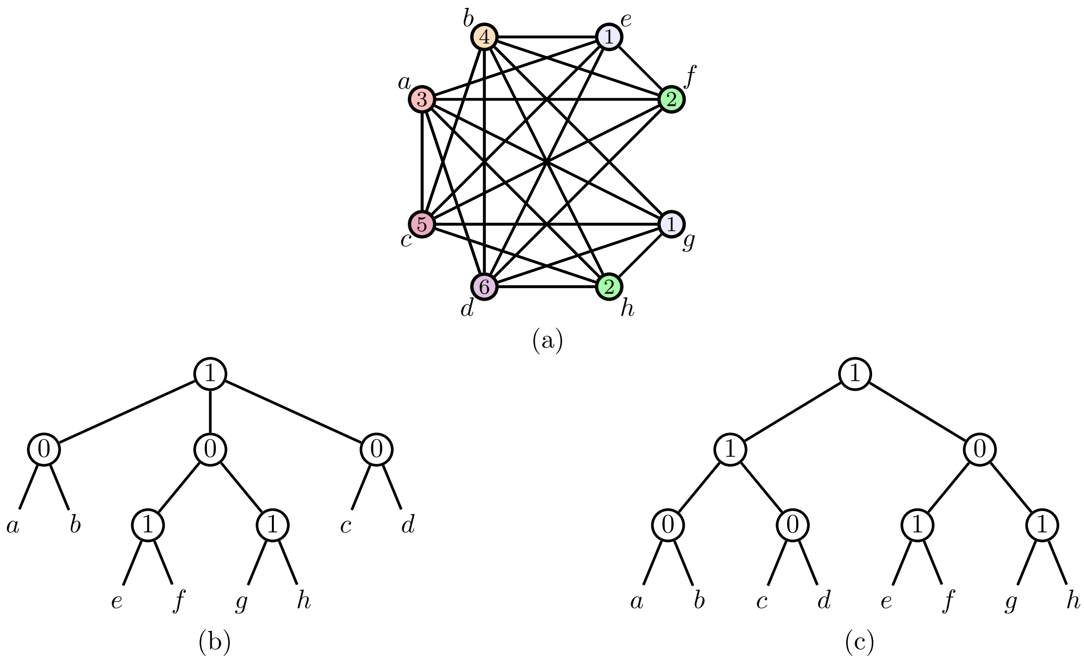

Figure 1: (a) A cograph G; (b) its canonical cotree; (c) a binary cotree. The graph G is shown along with an optimal acyclic coloring, which is (necessarily) also an optimal star coloring by Theorem 1.

G that uses $\chi_a(G)$ colors. We proceed in the following way, beginning with the root. As before, let t_i be a node in T_G with children t_j and t_k. If t_i is a 0-node, then we obtain an optimal acyclic coloring of G_i by combining optimal acyclic colorings of G_j and G_k, which are obtained recursively. If t_i is a 1-node, then assume without loss of generality that t_j is marked as saturated. We obtain an optimal acyclic coloring of G_i by first finding an optimal acyclic coloring of G_k (recursively, as before), and then coloring each vertex in V_j with a distinct color in $\{1 + \chi_a(G_k), \ldots, |V_j| + \chi_a(G_k)\}$. Correctness follows from Lemma 1. By Theorem 1, ϕ is also an optimal star coloring of G.

The number of steps taken by each of the two phases of the algorithm is proportional to the number of nodes in the cotree, which is $O(n)$. □

If G is not given along with a corresponding cotree, then one can be obtained in $O(n + m)$ time [10]. Note that the running time of our algorithm is thus $O(n + m)$ when this step is required.

3 Triangulating Acyclically Colored Graphs

A graph is *chordal* if it has no induced cycle on four or more vertices. A *triangulation* of a graph $G = (V, E)$ is a chordal graph $G^+ = (V, E^+)$ such that $E \subseteq E^+$. The *clique number* of a graph G, denoted $\omega(G)$, is the largest number of pairwise adjacent vertices in G. The *treewidth* of a graph G, denoted $tw(G)$, is the minimum value of $\omega(G^+) - 1$ over all triangulations G^+ of G.

Theorem 4 ([15, 17])**.** *If G is a chordal graph, then $\chi_a(G) = \chi(G) = \omega(G) = tw(G) + 1$.*

Theorem 5 (folklore)**.** *For any graph G, $\chi_a(G) \leq tw(G) + 1$.*

Proof. By the definition of treewidth, there exists a triangulation G^+ of G such that $\omega(G^+) = tw(G) + 1$. Since G^+ is chordal and since chordal graphs are perfect [17], G^+ further satisfies $\omega(G^+) = \chi(G^+)$. The claim then follows from the observation that every proper coloring of G^+ is an acyclic coloring of G. □

Let G be a graph given with a proper coloring ϕ. We say that G can be ϕ*-triangulated* if there is a triangulation G^+ of G such that ϕ is a proper coloring of G^+. Determining whether G can be ϕ-triangulated is **NP**-complete [4]. This is known as the *triangulating colored graphs* problem, and it is polynomially equivalent to the perfect phylogeny problem from molecular biology [20]. The following result characterizes the acyclic colorings ϕ of a cograph G as exactly those colorings for which G can be ϕ-triangulated.

Theorem 6. *If ϕ is a proper coloring of a cograph G, then ϕ is an acyclic coloring of G if and only if ϕ is a proper coloring of some triangulation of G.*

Proof. Sufficiency is established as in the proof of Theorem 5.

Let ϕ be an arbitrary acyclic coloring of G, and let T_G be any cotree of G. We will construct a triangulation G^+ of G such that ϕ is a proper coloring of G^+. Throughout this process, we will maintain the property that G is a cograph. We can therefore describe the transformation in terms of modifications in T_G. One can easily see that a cograph has an induced cycle on four or more vertices if and only if there is some 1-node in the cotree with distinct children t_1, t_2 such that the subtrees rooted at t_1 and t_2 each contain a 0-node. Let t be a 1-node in T_G with children $t_1, t_2, \ldots, t_k$. Since ϕ is an acyclic coloring of G, it follows from Lemma 1 that there is at most one child t_i of t such that ϕ uses fewer than $|V_i|$ colors. We now modify the cotree as follows. For every child t_j except t_i, make all the leaves of the subtree rooted at t_j children of t, and delete the subtree rooted at t_j. The result of applying this procedure to every 1-node in T_G is a new cotree T_{G^+}; since every 1-node of T_{G^+} has at most one child that is not a leaf, it follows that the corresponding graph G^+ is a triangulation of G. Since we add edges only between vertices that have distinct colors, it follows that ϕ is a proper coloring of G^+, which completes the proof. □

Corollary 7. *If G is a cograph, then $\chi_a(G) = tw(G) + 1$.*

Note that we can check in polynomial time whether ϕ is an acyclic coloring of G, and the procedure described in the proof of Theorem 6 can be used to obtain a compatible triangulation. Figure 2 illustrates this concept for the graph depicted in Figure 1(a).

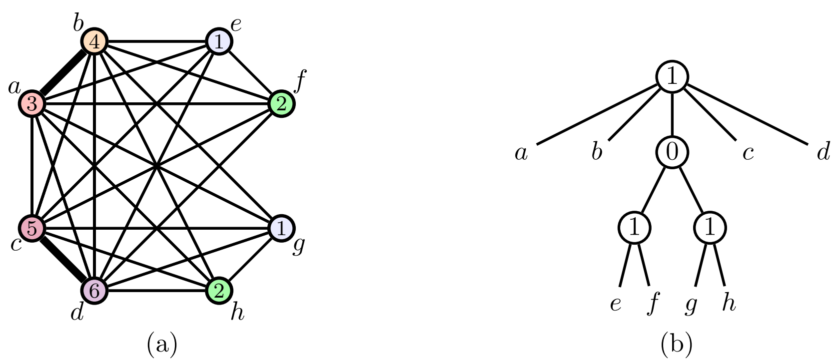

Figure 2: (a) A triangulation G^+ of the graph G depicted in Figure 1(a) (bold edges added during the triangulation process), which satisfies $\chi_a(G) = \chi(G^+) = \omega(G^+) = 6$; (b) the canonical cotree T_{G^+} corresponding to G^+.

A graph is an *interval graph* if its vertices can be put in correspondence with intervals on the real line such that two vertices are adjacent if and only if the corresponding intervals have a nonempty intersection. An *intervalization* of a graph $G = (V, E)$ is an interval graph $G^+ = (V, E^+)$ such that $E \subseteq E^+$. The *pathwidth* of a graph G, denoted $pw(G)$, is the minimum value of $\omega(G^+) - 1$ over all intervalizations G^+ of G. Note that since the interval graphs form a proper subclass of the chordal graphs, we have that $tw(G) \leq pw(G)$ for all graphs G. Bodlaender and Möhring obtained the following result by showing that every triangulation of a cograph G is also an intervalization of G.

Theorem 8 ([5])**.** *If G is a cograph, then $tw(G) = pw(G)$.*

Combining Corollary 2, Corollary 7, and Theorem 8 we obtain the following result.

Theorem 9. *If G is a cograph, then $\chi_s(G) = \chi_a(G) = tw(G) + 1 = pw(G) + 1$.*

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4 Concluding Remarks

Theorem 5 implies a natural heuristic for the acyclic coloring problem: simply find a triangulation G^+ of G that is close to optimal (with respect to treewidth), and then compute an optimal proper coloring of G^+, using $O(n+m)$ time [17]. Here we use the fact that treewidth is a particularly well-studied parameter, and there are many heuristics, approximation algorithms, exact (exponential) algorithms, and polynomial time algorithms for many classes of graphs [6, 13, 21]. In particular, for a constant k there is a linear-time algorithm for determining whether the treewidth of a graph is at most k and, if so, finding a corresponding triangulation [3].

Furthermore, Lemma 1 applies to any graph that is decomposable with respect to the join operation, and so it may be used as a reduction step that should be applied as the first step of any heuristic. Moreover, Lemma 1 implies that we can also find an optimal acyclic or star coloring of any graph for which these problems can be solved on all the graphs that result from recursively applying the join decomposition. For example, the *tree-cographs* [24] are those graphs that result by taking disjoint unions and joins of trees or other tree-cographs. The class of cographs is properly contained within this class. Since it is trivial to find an optimal acyclic or star coloring of a tree in linear time [12], it follows that we can solve these problems in linear time on the entire class of tree-cographs.

In the proof of Theorem 6, we were able to add at least one edge to every induced cycle on four vertices in G (which was given along with an acyclic coloring) such that no new induced cycles were created. However, one can easily construct an example for general graphs where this is not the case. Furthermore, there are graphs G with acyclic colorings ϕ for which G cannot be ϕ-triangulated. Two minimal examples are shown in Figure 3.

twin-C_5 co-twin-C_5

Figure 3: Two graphs, each given with acyclic coloring ϕ such that neither can be ϕ-triangulated. In the graph on the left, we cannot add an edge incident on d or a without creating a bichromatic cycle or violating the condition that the coloring is proper. Therefore, the cycles induced by $\{b,d,e,f,a\}$ and $\{c,d,e,f,a\}$ must be triangulated by adding edges $\{b,e\}$ and $\{c,f\}$, respectively. This results in $\{c,b,e,f\}$ inducing a bichromatic cycle. Note that edge $\{c,b\}$ must be added in any triangulation of the graph on the right, which reduces the problem to that of the graph on the left.

In Theorem 6 we proved the equivalence of the acyclic coloring and treewidth problems for cographs by showing that every acyclic coloring of a cograph G is a proper coloring of some triangulation of G. It would be useful to prove similar results for other classes of graphs; it is natural to consider other classes for which the treewidth problem can be solved in polynomial time.

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